

Closed k -strings in $SU(N)$ gauge theories : 2+1 dimensions

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Abstract

We calculate the ground state energies of closed k -strings in (2+1)-dimensional $SU(N)$ gauge theories, for $N = 4, 5, 6, 8$ and $k = 2, 3, 4$. From the dependence of the ground state energy on the string length, we infer that such k -strings are described by an effective string theory that is in the same bosonic universality class (Nambu-Goto) as the fundamental string. When we compare the continuum k -string tensions to the corresponding fundamental string tensions, we find that the ratios are close to, but typically 1% – 2% above, the Casimir scaling values favoured by some theoretical approaches. Fitting the N -dependence in a model-independent way favours an expansion in $1/N$ (as in Casimir scaling) rather than the $1/N^2$ that is suggested by naive colour counting. We also observe that the low-lying spectrum of k -string states falls into sectors that belong to particular irreducible representations of $SU(N)$, demonstrating that the dynamics of string binding knows about the full gauge group and not just about its centre.

1 Introduction

In this paper we calculate the ground state energy of a confining flux loop that winds once around a spatial torus of length l , with the flux in a higher representation than the fundamental. We do so for a number of $SU(N)$ groups in $D = 2 + 1$ dimensions. This study complements recent calculations for fundamental flux loops [1] and the excited state spectra of these [2]. In a forthcoming paper we shall analyse the excited state spectrum of multiply wound flux loops [3] and similar calculations in $D = 3 + 1$ dimensions are under way.

These numerical calculations have a number of motivations. Firstly there are old ideas that the relevant degrees of freedom of linearly confining $SU(N)$ gauge theories are string-like flux tubes and that these should be described by an effective string theory [4] which might describe all the important physics of the field theory, particularly as $N \rightarrow \infty$ [5]. Calculations such as ours can serve to test this programme; and, at the very least, to pin down the details of the effective string theory that describes the dynamics of long flux tubes [6, 7]. Secondly, there has been dramatic recent progress in the construction of string duals to various supersymmetric field theories [8]. There is a great effort to extend this to gauge theories and to QCD (see [9] for recent reviews). Our calculations may provide useful information in the search for the appropriate construction.

Our recent calculation [1, 2] of the spectrum of a closed loop of fundamental flux as a function of its length l , demonstrated that this spectrum is accurately given by the simple Nambu-Goto string model [10] down to lengths much smaller than those at which an expansion in powers of $1/\sigma l^2$ diverges. While this does not contradict the conventional approach of starting with a background consisting of a long ‘straight’ string and expanding in fluctuations around it [6], it does imply that one can do considerably better by expanding instead in small corrections around the full Nambu-Goto solution. Hints of this possibility can already be discerned in some recent work, [11, 12], based on [6, 7] respectively, that showed that the next term beyond the Luscher term [6] in the expansion in powers of $1/\sigma l^2$ is also universal and has the same coefficient as in the Nambu-Goto case. This fact, that we can identify the states as being essentially those of the Nambu-Goto model with the corresponding ‘phonon’ occupation numbers, means that the observed small deviations from Nambu-Goto have the potential to provide detailed and powerful constraints on the form of the additional string interactions that are needed.

The k -string that is the object of our study in this paper, can be thought of as a bound state of k fundamental strings. Earlier calculations have shown that for $N < \infty$ there do indeed exist such bound states both in $D = 2 + 1$ and in $D = 3 + 1$ [13, 14, 15, 16]. This binding cannot be readily incorporated into the usual analytic frameworks [6, 7] since the binding interaction presumably involves the exchange of small closed loops of string that are not under analytic control. Thus such strings are of particular interest.

In the case of $SU(N)$ gauge theories in $D = 2 + 1$ there is an additional motivation. There has been significant recent progress towards an analytic solution of these theories using a variational Hamiltonian approach [17]. (See also related work in [18].) These calculations make predictions for both the fundamental string tension and for k -string tensions. In [1] we found that the former prediction is very accurate and improves with increasing N , being only

about 1% off the correct value at $N = \infty$. (Although this discrepancy is statistically very significant.) In this paper we shall provide a similar test for the predicted k -string tensions.

In the next section we describe our method for calculating the energy of a closed k -string. We then present, in Section 3, our most accurate results for the l -dependence of this energy (which happens to be for the case $k = 2$) and compare to the expectation of the simple bosonic string model. In Section 4 we calculate the k -string tensions, extrapolate these to the continuum limit, and test the conjecture that they scale with the appropriate Casimir. We follow this, in Section 5, with a model-independent analysis of the N -dependence of these string tensions. Finally, in Section 6, we look at the wave-functions of the ground and excited states and show that in fact they fall into representations of the group, rather than being determined simply by the center. A summary of this work has been presented in [19].

2 Methodology

Consider a spatial torus of length l with all the other tori so large that they may be considered infinite. If the system is linearly confining, any flux around the torus will be confined to ‘string-like’ flux tubes. If the source from which such a flux would emanate transforms as $\psi \rightarrow z^k \psi$ under a gauge transformation, z , that belongs to the center, Z_N , of the $SU(N)$ gauge group, the flux tube is called a k -string. This is a useful categorisation since it is invariant under screening by gluons. For large l the ground state k -string energy $E_{0,k}(l)$ gives us the k -string tension,

$$\sigma_k = \lim_{l \rightarrow \infty} \frac{E_{0,k}(l)}{l}. \quad (1)$$

(In this paper σ_k is the tension of the ground state k -string, $\sigma_f \equiv \sigma_{k=1}$ is the fundamental string tension, $\sigma_{\mathcal{R}}$ is the tension of a string carrying flux in representation \mathcal{R} , and σ is a generic string tension. When there is no ambiguity, we drop the extra k label on E_0 .) Whether stable k -strings know about the full group or only about the center of the group, is one of the issues we address in this paper.

As one varies l there will be a finite volume phase transition at $l_c = 1/T_c$, where T_c is the value of the deconfining temperature. For $l \geq l_c$ the flux around the torus will be in a flux tube of length l and so we can ask what is the effective string theory that describes its spectrum. For $l < l_c$ there is no winding flux tube about which to ask such a question.

The alternative approach of studying open strings using Wilson loops (or pairs of un-smeared Polyakov loops), is complicated by the contribution of the sources to the corresponding ground state energy. The closed string setup provides a theoretically cleaner framework for studying the effective string theory although for smaller N , where the phase transition is second order or weakly first order, the influence of the nearby critical point needs to be taken into account once $l \sim l_c$. On the other hand open strings ending on point sources provide a way to address the important question of how perturbative physics at short distances evolves into non-perturbative physics at large distances, and whether there is a non-analyticity in this evolution as $N \rightarrow \infty$. Comparing the $SU(5)$ potential energy calculated in [20] with the

lower N potentials calculated in [21] points to the possibility of an interesting answer to this question.

Possible effective string theories fall into universality classes. If we expand $E_0(l)$ in powers of $1/\sigma l^2$ around σl , then for a bosonic string, where the only zero modes are those that arise from the $D-2=1$ transverse translations, the first two correction terms are universal [11, 12]

$$E_0(l) = \sigma l - \frac{c\pi}{6l} - \frac{c^2\pi^2}{72\sigma l^3} + \dots \quad ; \quad c = 1. \quad (2)$$

More generally, if there are additional zero modes along the string, then we may have $c \neq 1$ (although in that case only the first correction term has been shown to be universal). The simplest example of a bosonic string is the Nambu-Goto free string model, whose ground state energy is [10]

$$E_0(l) = \sigma l \left(1 - \frac{\pi}{3} \frac{1}{\sigma l^2} \right)^{\frac{1}{2}} \quad (3)$$

which reproduces eqn(2) when expanded in powers of $1/\sigma l^2$.

Our calculations are performed using standard lattice techniques. We work on periodic hypercubic $L_x \times L_y \times L_t$ lattices with lattice spacing a . The degrees of freedom are $SU(N)$ matrices, U_l , assigned to the links l of the lattice. The partition function is

$$\mathcal{Z}(\beta) = \int \prod_l dU_l e^{-\beta \sum_p \{1 - \frac{1}{N} \text{ReTr} U_p\}} \quad ; \quad \beta = \frac{2N}{ag^2} \quad (4)$$

where U_p is the ordered product of matrices around the boundary of the elementary square (plaquette) labelled by p and g^2 is the coupling, which in $D=2+1$ has dimensions of mass. This is the standard Wilson plaquette action and the continuum limit is approached by tuning $\beta = 2N/ag^2 \rightarrow \infty$. One expects that for large N physical masses will be proportional to the 't Hooft coupling $\lambda \equiv g^2 N$, and this is indeed what one observes [22]. So if we vary $\beta \propto N^2$ then we keep the lattice spacing a fixed in physical units.

We will consider a loop of flux that winds once around the x -torus, so that it is of length $l = aL_x$. A generic operator ϕ_l that couples to such a periodic flux loop is an ordered product of link matrices along a space-like curve that winds once around the x -torus, with the link matrices in the same representation as the flux. Correlations are taken in the t direction so that the energy of the loop is an eigenstate of the Hamiltonian (transfer matrix) defined on the xy space. Such a correlator may be expanded

$$C(t) \equiv \langle \phi_l^\dagger(t) \phi(0) \rangle = \sum_{n=0} |\langle n | \phi(0) | vac \rangle|^2 e^{-E_n t} \quad ; \quad E_i \leq E_{i+1} \quad (5)$$

where, if we have confinement, E_0 is the energy of the lightest flux loop. Since the fluctuations that determine the error in the Monte Carlo calculation of $C(t)$ may be expressed as a correlation function with a disconnected piece, the error is approximately independent of t while $C(t)$ itself decreases exponentially with t . Thus one needs operators ϕ_l that have a large projection onto the desired state, so that this state dominates $C(t)$ at small t where the statistical noise

is still relatively small. This can be achieved using standard smearing/blocking/variational techniques (see e.g. [22, 14]). If we denote a Polyakov loop in the fundamental $k = 1$ representation by l_p , then typical operators for k -strings will be $\phi_l = \text{Tr}\{l_p^{k-j}\}\text{Tr}\{l_p\}^j$ and we include all such operators in our variational basis, including their smeared/blocked versions.

Since $E_0(l)$ becomes large for large l , it can only be accurately calculated for modest values of $l\sqrt{\sigma}$ and so we need to know what are the important finite l corrections to the linear term, $E_0(l) \simeq \sigma l$ if we are to extract a reliable value for σ . This is even more important for heavier strings with $k > 1$, since these are heavier than the fundamental $k = 1$ string. In [1] it was shown that for the fundamental string, corrections to eqn(3) are extremely small, even very close to l_c and one can safely extrapolate using eqn(3) from, say, $l\sqrt{\sigma_f} \sim 3$. However there is no guarantee that this will continue to be the case for $k > 1$. Indeed we expect that at fixed k the binding will vanish as $N \rightarrow \infty$, and so in that limit, at any fixed l however large, the lightest k -string will actually be k fundamental strings, each satisfying eqn(2) [23]. Moreover, the fact that all the strings must share the same critical length, l_c , leads us to expect much larger deviations from eqn(3) at small l for $k > 1$. In addition there have been suggestions [24] that $k > 1$ strings do not belong to the bosonic string universality class and have $c > 1$. Thus it is important to determine the l -dependence of the ground state k -string and this is the issue to which we now turn.

3 Central charge and l dependence of k -strings

We have performed finite volume studies for $N = 2, 3, 4, 5, 6, 8$ at various values of the lattice spacing a . Here we shall present two of our most accurate studies for $k = 2$ strings. These are for SU(4) at $\beta = 32.0$ and SU(5) at $\beta = 80.0$. In units of the fundamental string tension, the lattice spacings correspond to $a\sqrt{\sigma_f} \simeq 0.2153$ and $a\sqrt{\sigma_f} \simeq 0.1299$ respectively.

In Table 1 we present the $k = 2$ ground state energy as a function of $l = aL_x$. We also present, for comparison, the corresponding energy of the fundamental $k = 1$ string. The critical value of l at which one loses the string is given by $l_c \sim 5a$ and $l_c \sim 8a$ for SU(4) and SU(5) respectively. Thus our range of closed string lengths extends down to nearly the minimum possible value.

There are many useful ways to analyse the l -dependence. Here we shall fit neighbouring values of l to a Nambu-Goto formula with an ‘effective’ central charge:

$$E_0(l) = \sigma l \left(1 - c_{eff} \frac{\pi}{3} \frac{1}{\sigma l^2} \right)^{\frac{1}{2}} \quad (6)$$

This will tell us what is the universality class of the effective string theory that describes $k = 2$ flux tubes; in particular if $\lim_{l \rightarrow \infty} c_{eff} = 1$ then this universality class is that of the simple bosonic string, just as it is for fundamental strings. In addition the rate of approach of $c_{eff}(l)$ to its asymptotic value will tell us something about the size of the corrections.

In Fig. 1 we plot the value of c_{eff} versus l for the SU(4) calculation. We show values for the $k = 2$ string and also, for comparison, for the fundamental $k = 1$ string. We express the length l in units of the string tension σ_k . While this is the appropriate variable to use, as we

see from eqn(6), it does mean that the scale in fixed physical units is slightly different for the $k = 1$ and $k = 2$ analyses. The horizontal error bars end at the two values of l used in the fits to eqn(6). In Fig. 2 we have a corresponding plot for SU(5).

We observe in Figs.1,2 quite strong evidence for the claim that at long distances the $k = 2$ flux tube behaves like a simple bosonic string, just like the fundamental flux tube. This contradicts the conjecture in [24] that $\lim_{l \rightarrow \infty} c_{eff} = \sigma_k / \sigma_f$. (This latter ratio is, from our fits, about 1.35 and 1.52 for SU(4) and SU(5) respectively.)

However it is clear that the corrections to Nambu-Goto are very much larger for $k = 2$ than for $k = 1$. Indeed, if we limited our analysis to $l\sqrt{\sigma_k} < 2.5$ we might be led to agree with the conjecture of [24]. One way to quantify the size of the deviation is to add to eqn(3) a correction term that is of higher order than the universal terms in eqn(2), i.e.

$$E_0(l) = \sigma l \left(1 - \frac{\pi}{3} \frac{1}{\sigma l^2} + c' \frac{1}{(\sqrt{\sigma} l)^5} \right)^{\frac{1}{2}}. \quad (7)$$

Fitting, for example, our SU(5) loop masses gives

$$c' = \begin{cases} -0.09(2) & : & k = 1 \\ -0.93(5) & : & k = 2 \end{cases} \quad (8)$$

What is interesting about this result is that we see that it is not the $k = 2$ string that has an anomalously large non-universal correction, but rather it is the $k = 1$ string that has an anomalously small non-universal correction. A value $c' = O(1)$ is precisely what one might expect for a generic effective string model in the Nambu-Goto universality class and that is what we find for the $k = 2$ flux tube. This observation appears to be both insensitive to the value of N and to the size of a : that is to say, it applies to the continuum limit.

Although our calculations of $E_0(l)$ for $k > 2$ are not accurate enough to provide evidence as compelling as for $k = 2$, they are consistent with $k > 2$ strings also belonging to the Nambu-Goto universality class, and so we shall assume that to be the case. Within our statistical errors it also appears justified to assume that the dominant finite- l corrections are those in eqn(3).

4 Continuum string tensions and Casimir scaling

The prediction of [17] is that string tensions in D=2+1 SU(N) gauge theories should be proportional to the quadratic Casimir of the representation of the flux. This is, of course, the exact result in D=1+1, since in that case linear confinement is the Coulomb interaction. Based on the idea of effective dimensional reduction driven by a highly disordered vacuum, it was conjectured a long time ago [25] that this might also hold in $D = 2 + 1$ and $D = 3 + 1$. There is some additional evidence for this hypothesis from calculations of the potential between charges in various representations of SU(3) [26].

For a given k the smallest Casimir arises for the totally antisymmetric representation, and this should therefore provide the ground state k -string tension:

$$\frac{\sigma_k}{\sigma_f} = \frac{k(N-k)}{N-1}. \quad (9)$$

This is the part of the ‘Casimir Scaling’ hypothesis that we shall be mainly testing in this paper. For this purpose it is useful to have an alternative conjecture that possesses the correct general properties. A convenient and well-known example is provided by the trigonometric form

$$\frac{\sigma_k}{\sigma_f} = \frac{\sin \frac{k\pi}{N}}{\sin \frac{\pi}{N}} \quad (10)$$

that was originally suggested on the basis of an M-theory approach to QCD [27].

In fact the full prediction of [17] for σ_k is more specific than eqn(9), since it also predicts a value for σ_f in terms of g^2 , and including this gives:

$$\frac{\sigma_k}{(g^2 N)^2} = \frac{k(N-k)(N+1)}{N^2} \frac{1}{8\pi}. \quad (11)$$

Since we already know that the prediction for the $k=1$ (fundamental) string tension is too high by about 1 – 2% (depending on N) [1], testing eqn(11) and eqn(9) does not come to exactly the same thing.

We now turn to our calculations of σ_k . We will illustrate these using the example of SU(5) and then state our results for other values of N .

The first step is to extract ground state energies from the Euclidean correlation functions that we obtain as a result of the variational calculation described in Section 2. The standard method is to fit the ground state correlator with a single exponential, $|c|^2 \exp\{-E_0(l)t\}$, for $t \geq t_0$, and to find the minimum value of t_0 that provides an acceptable fit. (Usually the upper limit of the range in t is chosen so as to exclude the very noisy values at large t .) The value of the ground state energy obtained by such single-exponential fits we label by S . Of course, since in general $|c|^2 < 1$, we know that there must be some contribution even at $t \geq t_0$ from the excited states in eqn(5). In the range of t fitted this contribution will be masked by the statistical errors, but will lead to a systematic downward shift in all extracted values of $E_0(l)$. Ideally this shift should be within the statistical errors but this is far from guaranteed given our inevitably imperfect control of the statistical analysis. To bound this systematic error, we also perform fits with two exponential terms where the energy of the excited state is chosen to be that of the first excited state (using S fits) with the same quantum numbers. This will usually provide an upper bound on the excited state correction to the ground state energy. We label the ground state energy obtained by such double-exponential fits by D . (Note that this choice will not necessarily bound the effect of such contributions to the *ratio* of string tensions.)

We calculate the values of $E_0(l)$ for strings of various k on lattices with spatial tori that satisfy $l\sqrt{\sigma_f} > 3$. We have seen in Section 3 that at such distances the Nambu-Goto expression in eqn(3) accurately represents the l -dependence of the ground state string energy (within

statistical errors that are typical of the present calculation). We therefore use eqn(7) to extract values of σ_k from the corresponding $E_0(l)$.

Having obtained values of σ_k at various values of β we now want to extrapolate the dimensionless ratios σ_k/σ_f to the continuum limit. We do so using:

$$\frac{\sigma_k(a)}{\sigma_f(a)} = \frac{\sigma_k(0)}{\sigma_f(0)} + c_1 a^2 \sigma_f + c_2 a^4 \sigma_f^2 + \dots \quad (12)$$

where the c_i can be treated as constants over our range of β . Typically extrapolations of this kind include just the $O(a^2)$ correction, and one drops the points at the largest values of a until a statistically acceptable fit is possible. We shall also perform a $O(a^4)$ fit so as to estimate the systematic error in the continuum extrapolation.

We show the resulting values of σ_k/σ_f in SU(5) in Figs.3,4, where the calculations have been performed using single and double exponential fits respectively. We also show linear $O(a^2)$ and quadratic $O(a^4)$ extrapolations to the continuum limit in each case. The resulting continuum values, together with the values of χ^2 per degree of freedom for the best fits, are listed in Table 2. Comparing Fig.3 with Fig.4 we see that the S and D values are quite consistent although the errors of the latter are significantly larger. The corresponding continuum limits are, as a result, also very similar. If we now compare the two different continuum extrapolations, we see that including an extra $O(a^4)$ correction reduces the continuum value by a little more than one standard deviation. In principle the fit with the extra correction term should be more reliable. However the coefficient of such a term is in practice largely determined by the calculations at the largest values of a , where there is always the danger that one is being influenced by the strong-to-weak coupling crossover where the power series expansion in a^2 breaks down. In any case, we see from this study that neither the use of double exponential fits, nor the inclusion of higher order terms in the continuum extrapolation, makes much difference to the final result. Our qualitative conclusion is that while the continuum ratio is close to the Casimir Scaling value of 1.5, there is a significant and robust discrepancy at the $\sim 1.5\%$ level.

We have repeated the above calculation for SU(4), SU(6) and SU(8) and the resulting continuum values, using $O(a^2)$ extrapolations, are presented in Table 3. The first error is statistical and the second is intended to give some idea of the maximum possible systematic error, by looking at what one obtains from various combinations of S and D fits to σ_k and σ_f . (As such it is not an estimate of the actual systematic error which would undoubtedly be significantly smaller.) Some remarks. Firstly, for SU(6) our best fit has a rather poor, but not impossible, χ^2 . We have therefore doubled the errors so as to achieve a reasonable confidence level. This will ensure that the $N = 6$ value does not distort our analysis of the N -dependence in the next Section. Second remark: for $N = 8$ one of the two finite volume calculations has a very poor χ^2 when we attempt to extract the string tensions. We therefore do not include the resulting values, that have very small errors but very small confidence levels, in our continuum extrapolations, but instead take from that study the string tension as calculated from the lattice whose size (in physical units) is the same as we use at other values of β . (Some freedom as to which values to include at those β at which we perform finite volume studies, as well as some freedom in the fitting process, explains why the SU(5) values

in Table 3 do not coincide exactly with those in Table 2.) Note that as k increases, the loop energy also increases, and the errors become less reliable. Nonetheless, despite these caveats, we see from Table 3 that we can confidently conclude that while the string tension ratios are remarkably close to the Casimir Scaling values, and much further away from (MQCD) Sine Scaling, there is a definite discrepancy that is typically at the 1 – 2% level.

It is interesting to note that if we take the actual values of $\sqrt{\sigma_f/g^2N}$ as obtained in [1] together with the values of σ_k in eqn(11), we obtain the values of σ_k/σ_f listed in the last column of Table 3. We see that these values are largely consistent with our calculated numbers. That is to say, while the prediction of [17] for σ_f is too high by $\sim 1-2\%$, the prediction for $\sqrt{\sigma_k/g^2N}$ with $k > 1$ is actually consistent with the values we obtain. This is particularly significant for the $k = 2$ values which are the ones that we determine most accurately and reliably.

5 N -dependence

An interesting feature of the Casimir Scaling hypothesis in eqn(9) is that if we expand the expression for σ_k/σ_f in powers of $1/N$ at fixed k , we see that the leading correction is $O(1/N)$ rather than the $O(1/N^2)$ that one would expect from the usual colour counting rules. This observation [28] has generated some controversy [29, 16]. It is therefore interesting to see if we can learn something about the power of the leading correction from a more model-independent analysis of our results.

We begin with our results for the $k = 2$ continuum string tensions, as these are more accurate than those for larger k . We attempt a fit with the ansatz

$$\frac{\sigma_{k=2}}{\sigma_f} = 2 - \frac{a}{N^p} - \frac{b}{N^{2p}} \quad (13)$$

using the values $p = 1$ and $p = 2$, and show the best such fits in Fig.5.

For $p = 1$ our best fit gives $a = 1.48(5)$ and $b = 4.42(21)$, with $\chi^2/n_{df} \simeq 2.7/2 = 1.35$. Thus we infer that an $O(1/N)$ correction is quite consistent with our $k = 2$ calculations even if, as we see in Fig.5, the full Casimir Scaling prediction is not. It is also interesting to note that this best fit is consistent with the trivial value of $\sigma_{k=2}/\sigma_f = 1$ for SU(3), even though this value has not been included in the fit. For $p = 2$, on the other hand, we find no acceptable fit at all, and in order to proceed we have to drop the $N = 4$ point from the fit. We now only have one degree of freedom, and although we obtain an acceptable fit, with $a = 18.0(5)$ and $b = -155(12)$, for this reduced dataset, we see from Fig.5 that it has a wavy behaviour that is characteristic of an inappropriate functional form being forced onto data using large cancellations between different terms. This perception is reinforced when we note how much this fit deviates from the SU(4) value. We conclude from all this that our $k = 2$ results strongly disfavour a conventional $O(1/N^2)$ correction.

Complementary to the above large N limit, in which k is kept fixed, is the limit where k/N is held fixed. The extreme example is $k = N/2$; i.e. $k = 2, 3, 4$ for $N = 4, 6, 8$ respectively. Of course the $k = 4$ string is quite massive, so it will have a large statistical error, and quite

possibly a large systematic error as well. Proceeding anyway, we fit with the ansatz

$$\frac{\sigma_{k=N/2}}{\sigma_f} = \frac{N}{2} \left\{ a + \frac{b}{N^p} \right\} \quad (14)$$

with $p = 1, 2$. As we see in Fig.6, the best $p = 1$ fit is acceptable (with $\chi^2 \simeq 1.5$ for 1 degree of freedom) and gives $a = 0.501(4)$ and $b = 0.70(2)$. It is interesting to note that this value of a is consistent with what one obtains at $N = \infty$ from Casimir scaling, although, as we see in Fig.6, the latter is actually far from the calculated ratios at finite N . The $p = 2$ best fit is noticeably worse, with $\chi^2/n_{df} \simeq 2.0$ and giving $a = 0.567(3)$ and $b = 1.78(5)$, but it is not so bad that it can be completely excluded. Nonetheless it mildly reinforces our earlier conclusion that $O(1/N^2)$ corrections are disfavoured.

6 Excited strings : group or centre?

As we have seen (here and in earlier work) there do indeed exist non-trivial k -string bound states. The ground state k -string is stable, with string tension $\sigma_k < k\sigma_f$ at finite N , and we have provided strong evidence that it falls into the universality class of the simple Nambu-Goto string model. Thus one may conjecture that there will be a spectrum of Nambu-Goto-like excitations of this k -string, at long distances, for which the scale is set by σ_k .

We also found that the string tension of the ground state k -string is remarkably close to the prediction of Casimir Scaling i.e. that it is proportional to the quadratic Casimir of the totally antisymmetric representation of k fundamentals. If this is more than mere coincidence, it implies that we should also expect to see some sign of flux tubes corresponding to other representations, with string tensions (approximately) proportional to the corresponding Casimirs. Of course, since these Casimirs are larger, such flux tubes will not be stable, but can be screened by gluons to the totally antisymmetric one. If the amplitude for such screening is large, such ‘resonant’ states will be ‘broad’, difficult to identify, and will presumably play no significant dynamical role. However if the amplitude for such screening is small enough, then it should be possible to identify such ‘resonant’ flux tubes in the excited state spectrum of the k -strings. Indeed, in this case, one may ask if such a ‘resonant’ flux tube possesses its own spectrum of Nambu-Goto-like excitations.

By contrast, in the free Nambu-Goto string model, which has been found to work very well for fundamental strings that wind once around a torus, [1, 2] there are of course no bound state k -strings; we simply have $\sigma_k = k\sigma_f$ (as indeed one expects to have in the $SU(N)$ gauge theory at $N = \infty$). One may speculate that the observed binding can be encoded through some correction terms to the Nambu-Goto action, leading to a bound state string for each value of k . Moreover it is possible that each of these stable bound states will be accompanied by its own tower of Nambu-Goto-like excited states, once the string is long enough. However, while there may be additional ‘resonant’ string excitations, it is not clear how these correction terms could embody the detailed group structure of the $SU(N)$ gauge theory in any natural fashion, and so one would not expect the excited states to be related in any way to the different representations of the $SU(N)$ gauge group.

There is some evidence from calculations of potentials between sources in different representations that the corresponding fluxes do form flux tubes [26], but such calculations do not go to large enough separations to be unambiguous. There is also some evidence from earlier calculations of closed loops, that the lowest excited $k = 2$ strings fall approximately into symmetric and anti-symmetric representations [13], although the observed near-degeneracies in these states [14] have been interpreted as saying that these are in fact the same states, and that the categorisation at a given k into different representations is illusory: that is to say, confining strings only know about the center of the group [30].

To determine what is actually the case, one needs accurate calculations of a significant number of excited states at each k . This requires a large basis of operators enabling one to obtain good overlaps onto these excited states, paralleling the $k = 1$ calculations in [1]. Such a calculation is under way [3]. In the meantime, it is interesting to see what we can extract from the present calculation which has a rather limited basis, with overlaps on to the excited k -string states that are only moderately good.

We restrict ourselves here to the $k = 2$ excited state spectrum which is the one that we can determine most accurately. The operators whose correlators we have calculated are of the form $\text{Tr} l^2$ and $\{\text{Tr} l\}^2$ where l is a Polyakov loop. There will be such operators corresponding to each of the smearing/blocking levels and this provides us with a non-trivial basis for our variational calculation of the $k = 2$ spectrum. This basis of operators allows us to construct Polyakov loops in the totally symmetric (2S) and antisymmetric (2A) representations:

$$\text{Tr}_{2A} l = \text{Tr} l^2 - \{\text{Tr} l\}^2 \quad ; \quad \text{Tr}_{2S} l = \text{Tr} l^2 + \{\text{Tr} l\}^2. \quad (15)$$

To construct $k = 2$ operators in other representations would require higher powers, e.g. $\text{Tr} l^3 \text{Tr} l^\dagger$, which we have not calculated here. Thus we shall limit our analysis to the $k = 2A$ and $k = 2S$ representations. Using the basic operators in eqn(15), we construct $\vec{p} = 0$ operators at time t by summing up $\text{Tr}_{2A,2S} l$ over the spatial coordinates. We call the resulting operators $\Phi_{2A}(t)$ and $\Phi_{2S}(t)$ where there is an additional suppressed label, b , for the blocking level of the gauge links used in the construction of l . These are our basis operators and our variational procedure will (ideally) produce the linear combinations that are closest to the actual energy eigenstates.

One can immediately learn something interesting from these basis operators. Consider the normalised overlap of a symmetric operator Φ_{2S} at blocking levels b_S with an antisymmetric operator Φ_{2A} at blocking levels b_A :

$$O_{AS}(b_A, b_S) = \frac{\langle \Phi_{2A}^\dagger(0) \Phi_{2S}(0) \rangle}{\langle \Phi_{2A}^\dagger(0) \Phi_{2A}(0) \rangle^{1/2} \langle \Phi_{2S}^\dagger(0) \Phi_{2S}(0) \rangle^{1/2}}. \quad (16)$$

If these overlaps turn out to be large, then the screening/mixing effects are large, and there is not much point in trying to find different string states corresponding to different representations. If they are small, then it becomes interesting to determine the representation content of the ground and excited k -string states.

In Table 4 we list the values of these overlaps as calculated on a $24^2 32$ lattice in $\text{SU}(5)$, at a value of β that corresponds to $a\sqrt{\sigma_f} \simeq 0.13$. This is both close to the continuum limit

and ensures that the string is long enough, $l\sqrt{\sigma_f} \simeq 3.1$. Our results show that the overlaps are remarkably small. Indeed they are all consistent with zero, within very small errors, except for those that involve the very largest blocking level. Blocked links at this level spread right across the spatial volume and we have convincing evidence from calculations on larger volumes that the somewhat enhanced overlap is primarily a finite volume effect. Nonetheless even these non-zero values are in fact very small. This provides striking evidence that the screening dynamics is a weak perturbation on the classification of states into representations of $SU(N)$.

We now turn to our variational estimates of the $k = 2$ string eigenstates, Ψ_J , that we obtain using the full basis of $\Phi_{2A}(t)$ and $\Phi_{2S}(t)$ operators. We project the resulting eigenstates onto the $k = 2A$ and $k = 2S$ subspaces. (Which we do by forming an orthogonal basis out of our non-orthogonal $\Phi_{2A,2S}$ operators.) We call the corresponding overlaps O_{JA} and O_{JS} , where we expect $|O_{JA}|^2 + |O_{JS}|^2 \simeq 1$ since the $k = 2A$ and $k = 2S$ subspaces are nearly orthogonal. If the subspaces were exactly orthogonal then it would of course immediately follow that the eigenstates would fall into exact $k = 2A$ and $k = 2S$ representations. What we find is that they appear to do so to quite a good approximation. An example is shown in Fig.7. Here we plot the overlaps $|O_{JA}|^2, |O_{JS}|^2$ for the first five $k = 2$ states, $J = 1, \dots, 5$, as a function of the string length l , in a calculation in $SU(6)$ at $\beta = 59.4$ (which corresponds to $a\sqrt{\sigma_f} \simeq 0.28$). What we see is that the states are almost entirely either pure $k = 2A$ or $k = 2S$ except at certain values of l where the state changes from one representation to the other. Since this always happens to pairs of states at the same value of l and involves an opposite change in representation in this pair, it is clear what is happening: as we increase l the two energy levels cross at that l . So, for example, in Fig.7 the states that are Ψ_2 and Ψ_3 for $l\sqrt{\sigma_f} < 3$ exchange their ordering in energy for $l\sqrt{\sigma_f} > 4$. And a similar exchange occurs for Ψ_4 and Ψ_5 , but at a larger value of l . What we see, therefore, is that the k -string states belong to particular $SU(N)$ representations to a very good approximation. The apparent near-degeneracies observed in [14] are in fact accidental: they arise from the (natural) fact that the various excited states have energies that vary differently with l , so that they cross as l increases and when they do so we have degeneracies. As we have just seen, the first $k = 2$ crossing occurs at $l\sqrt{\sigma_f} \sim 3.5$ which is precisely where such degeneracies have been previously noted to occur [13, 14]. It is unambiguous that what we are seeing here is not a single state, as suggested in [30], but rather two that are nearly degenerate at appropriate values of l .

If we focus on the lightest three states in Fig.7, we see that the ground state is always totally anti-symmetric, while the first excited state is initially symmetric and then, for $l\sqrt{\sigma_f} > 4$, antisymmetric. This very different dependence on l between the ground state $k = 2S$ state and the first excited $k = 2A$ state is easily understood if we think of there being two separate Nambu-Goto towers of states labelled by the $k = 2A$ and $k = 2S$ representations respectively. In the Nambu-Goto model the energy levels would be given by

$$E_n(l) = \sigma_{\mathcal{R}} l \left(1 + 8\pi \frac{\pi}{\sigma_{\mathcal{R}} l^2} \left(n - \frac{D-2}{24} \right) \right)^{\frac{1}{2}} \quad (17)$$

where $\sigma_{\mathcal{R}}$ is the string tension of the flux in representation \mathcal{R} and n is an integer that counts the ‘phonon’ excitations along the string. For the ground string state we have $n = 0$ and

$E_0(l)$ increases approximately linearly with l in the range of l relevant here. On the other hand the first excited state with $n = 1$ will clearly have a very different variation with l . This is illustrated in Fig.8 where we plot the Nambu-Goto energy levels (divided by $\sigma_f l$) for towers of states in the $k = 2A$ and $k = 2S$ representations of $SU(6)$ using the string tensions that are predicted by Casimir Scaling. One sees how the very different l -dependence of the ground state $k = 2S$ string and the first excited $k = 2A$ string ensures that they cross for a value of $l\sqrt{\sigma_f}$ close to where we observe the crossing in Fig.7. Our estimates of the actual energy levels are very roughly consistent with such a scenario, although a reliable and precise comparison must await a dedicated study of the $k > 1$ energy spectrum [3]. In the meanwhile what we have is very good evidence that the k -string states fall, to a good approximation, into representations of $SU(N)$, and strong indications, from the crossings of the states, that there exists a ground state flux tube in representations such as $k = 2S$ where it is not completely stable.

7 Conclusions

In this paper we have studied k -strings that wind around a spatial torus of length l . This setup is convenient because it involves no sources so one knows that, for $l \geq 1/T_c$, all that one has is a closed string-like flux tube. It provides a clean way to determine the effective string theory describing the flux tube. Once this question has been addressed the second, equally important question of how the effective string theory at long distances matches onto short-distance asymptotic freedom, can be conveniently addressed through the calculation of the potential between sources as a function of the distance between these sources. For example, one can ask whether the running coupling in the ‘potential scheme’

$$V(r) \equiv -C \frac{\alpha_v(r)}{r} \quad (18)$$

(C is the quadratic Casimir for the representation of the sources) develops a non-analyticity at some critical distance as $N \rightarrow \infty$.

The main conclusions of our study of such closed k -strings are as follows.

- We find that just like the fundamental $k = 1$ flux tube, $k = 2$ flux tubes are described by an effective string theory that is in the universality class of the Nambu-Goto model. If one parameterises the correction to Nambu-Goto by an appropriate power of $1/\sigma l^2$, one finds that the coefficient is $O(1)$; that is to say, it is neither small nor large. This is in contrast to the case of the fundamental string, where such corrections are remarkably small [1].
- The $k \geq 3$ flux tubes are also consistent with being in the simple bosonic string universality class. While the statistical errors are larger than for $k = 2$, they also exclude a central charge $\propto \sigma_k/\sigma_f$ as suggested in [24].
- The ratio of the ground state string tensions, σ_k/σ_f , is within 1 – 2% of Casimir Scaling, but this discrepancy is statistically significant, and it is robust against systematic errors. On the other hand, the values of $\sigma_k/g^2 N$ are consistent with the predictions of [17] for $k > 1$; in particular for $k = 2$ where we have very precise results. From this point of view, the small

breakdown of Casimir Scaling that we observe can be entirely attributed to the fact that the predicted value of σ_f in [17] is slightly higher than its actual value as obtained in [1]. One should perhaps not read too much significance into this agreement. The important point is that any discrepancy between the predictions for σ_k in [17] and the true values are no larger for $k > 1$ strings than they are for the fundamental string.

- An analysis of how the $k = 2$ string tension varies with N strongly suggests that the corrections to the $N = \infty$ limit, $\sigma_{k=2} = 2\sigma_f$, come in powers of $1/N$ (as in Casimir Scaling) rather than in powers of $1/N^2$, as suggested by standard colour counting arguments. There is also some statistically weaker evidence for this from our analysis of the N -dependence of $\sigma_{k=N/2}$ for $N = 4, 6, 8$.

- The k -string eigenstates fall, very accurately, into representations of $SU(N)$. That is to say, k -strings know about the full group representation of the flux, and not just about its transformation properties under the centre. The near-degeneracies previously observed between some states assigned to different representations, should be interpreted as level crossing (as a function of the string length l). The intriguing possibility that stable k -strings (and perhaps even unstable ‘resonant’ strings) may possess their own towers of Nambu-Goto states, receives some support from preliminary precision results on the k -string spectrum [3].

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References

- [1] B. Bringoltz, M. Teper Phys. Lett. B645 (2007) 383 [hep-th/0611286].
- [2] A. Athenodorou, B. Bringoltz, M. Teper Phys. Lett. B656 (2007) 132 [arXiv:0709.0693].
- [3] A. Athenodorou, B. Bringoltz, M. Teper in preparation.
- [4] A. M. Polyakov, *Gauge Fields and Strings* (Harwood, 1987).
- [5] G. ’t Hooft, Nucl. Phys. B72 (1974) 461; B75 (1974) 461. J. Polchinski, hep-th/9210045.
- [6] M. Luscher, K. Symanzik and P. Weisz, Nucl. Phys. B173 (1980) 365.
- [7] J. Polchinski and A. Strominger, Phys. Rev. Lett. 67 (1991) 1681.

- [8] J. Maldacena, Adv.Theor. Math. Phys. 2 (1998) 231 [hep-th/9711200].
O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 (2000) 183 [hep-th/9905111].
- [9] K. Peeters and M. Zamaklar, arXiv:0708.1502.
J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall arXiv:0711.4467.
- [10] J. Arvis, Phys. Lett. 127B (1983) 106.
- [11] M. Lüscher and P. Weisz, JHEP 0407 (2004) 014 [arXiv:hep-th/0406205].
- [12] J. Drummond, arXiv:hep-th/0411017; arXiv:hep-th/0608109.
N. Hari Dass and P. Matlock, arXiv:hep-th/0606265, arXiv:hep-th/0611215.
- [13] B. Lucini, M. Teper Phys. Rev. D64 (2001) 105019 [hep-lat/0107007]; Phys. Rev. D66 (2002) 097502 [hep-lat/0206027]; Phys. Lett. B501 (2001) 128 [hep-lat/0012025].
- [14] B. Lucini, M. Teper, U. Wenger JHEP 0406 (2004) 012 [hep-lat/0404008].
- [15] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, Phys. Rev. D65 (2002) 021501 [hep-th/0106185]; JHEP 0201 (2002) 009 [hep-th/0111090].
- [16] C. Korthals Altes and H. Meyer, hep-ph/0509018.
- [17] D. Karabali, C. Kim and V. P. Nair, Phys. Lett. B434 (1998) 103 [hep-th/9804132];
V. P. Nair, Mod. Phys. Lett. A18 (2003) 2415; hep-th/0309061.
A. Agarwal, D. Karabali, and V. P. Nair, Phys. Rev. D77 (2008) 025014 [arXiv:0705.2898].
- [18] R. Leigh, D. Minic, A. Yelnikov Phys. Rev. D76 (2007) 065018 [hep-th/060406].
L. Freidel, R. Leigh, D. Minic, A. Yelnikov, arXiv:0801.1113.
R. Leigh and D. Minic, hep-th/0407051.
J. Greensite and S. Olejnik, arXiv:0709.2370; arXiv:0709.2370.
P. Orland, arXiv:0710.3733; Phys. Rev. D75 (2007) 101702 [arXiv:0704.0940]; Phys. Rev. D71 (2005) 054503 [hep-lat/0501026].
- [19] B. Bringoltz and M. Teper, PoS(LATTICE 2007)291 [arXiv:0708.3447].
- [20] H. Meyer, Nucl. Phys. B758 (2006) 204 [hep-lat/0607015].
- [21] N. Hari Dass, P. Majumdar hep-lat/0702019; arXiv:0709.4170;
P. Majumdar hep-lat/0406037.
- [22] M. Teper, Phys. Rev. D59 (1999) 014512 [hep-lat/9804008].
- [23] H. Meyer and M. Teper, JHEP 0412 (2004) 031 [hep-lat/0411039].
- [24] P. Giudice, F. Gliozzi, S. Lottini PoS (LAT2006) 065 [hep-lat/0609055]; PoS(LATTICE 2007)302 [arXiv:0710.0764].
M. Caselle, P. Giudice, F. Gliozzi, P. Grinza, S. Lottini, arXiv:0710.3522.

- [25] J. Ambjorn, P. Olesen and C. Peterson, Nucl. Phys. B240 (1984) 189, 533; B244 (1984) 262; Phys. Lett. B142 (1984) 410.
- [26] S. Deldar, Phys. Rev. D62 (2000) 034509 [hep-lat/9911008]; JHEP 0101 (2001) 013 [hep-ph/9912428].
G. Bali, Phys. Rev. D62 (2000) 114503 [hep-lat/0006022].
- [27] A. Hanany, M. Strassler and A. Zaffaroni, Nucl. Phys. B513 (1998) 87 [hep-th/9707244].
M. Strassler, Nucl. Phys. Proc. Suppl. 73 (1999) 120 [hep-lat/9810059].
M. Strassler, Prog. Theor. Phys. Suppl. 131 (1998) 439 [hep-lat/9803009].
- [28] M. Teper, Proceedings from the INT vol 12 (Ed. R. F. Lebed, World Scientific 2002) [hep-ph/0203203].
- [29] A. Armoni and M. Shifman, Nucl. Phys. B671 (2003) 67. [hep-th/0307020]; Nucl. Phys. B664 (2003) 233 [hep-th/0304127].
- [30] A. Armoni and B. Lucini, JHEP 0606 (2006) 036 [hep-th/0604055]. Nucl.Phys.B664:233-246,2003.

SU(4) $\beta = 32.0$			SU(5) $\beta = 80.0$		
L_x	$aE_0(k=1)$	$aE_0(k=2)$	L_x	$aE_0(k=1)$	$aE_0(k=2)$
6	0.1637(8)	0.2428(17)	10	0.1015(6)	0.1720(20)
8	0.2976(6)	0.4241(11)	12	0.1519(5)	0.2485(12)
10	0.4075(5)	0.5716(10)	16	0.2344(5)	0.3747(9)
12	0.5105(7)	0.7043(23)	20	0.3089(6)	0.4867(11)
16	0.7073(13)	0.9656(35)	24	0.3827(6)	0.5938(15)
			32	0.5242(11)	0.8090(28)

Table 1: The masses of the lightest $k=1$ and $k=2$ flux loops as a function of their length $l = aL_x$, for SU(4) and SU(5) at the indicated values of $\beta = 2N/g^2$.

$\lim_{a \rightarrow 0} \sigma_{k=2}/\sigma_f$; SU(5)				
fit	S	χ^2/n_{df}	D	χ^2/n_{df}
$O(a^2)$	1.528(4)	1.3	1.524(7)	0.9
$O(a^4)$	1.520(5)	1.1	1.509(11)	0.7

Table 2: Continuum limits of the extrapolations in Figs3,4 with quality of best fit.

k	N	σ_k/σ_f	Casimir	Sine	Nair
2	4	$1.3552(22) \binom{91}{29}$	1.333..	1.414..	1.361
2	5	$1.5275(26) \binom{96}{0}$	1.5	1.618..	1.529
2	6	$1.6234(66) \binom{98}{0}$	1.6	1.732..	1.629
2	8	$1.7524(51) \binom{107}{30}$	1.714..	1.848..	1.741
3	6	$1.8522(48) \binom{134}{0}$	1.8	2.0	1.832
3	8	$2.174(19) \binom{14}{9}$	2.143..	2.414..	2.177
4	8	$2.366(11) \binom{11}{6}$	2.286..	2.613..	2.322

Table 3: k -string tensions compared to Casimir and Sine Scaling conjectures in eqns(9,10) and the Nair prediction in eqn(11). First error statistical, second estimate of systematics.

b_A	b_S	O_{AS}	b_A	b_S	O_{AS}	b_A	b_S	O_{AS}
1	1	-0.0003(8)	2	1	0.0006(6)	3	1	0.0006(5)
1	2	-0.0005(8)	2	2	0.0005(7)	3	2	0.0005(7)
1	3	-0.0006(8)	2	3	0.0002(8)	3	3	-0.0001(10)
1	4	0.0002(8)	2	4	0.0006(10)	3	4	0.0008(12)
1	5	0.0025(6)	2	5	0.0047(8)	3	5	0.0066(11)
4	1	0.0006(6)	5	1	-0.0002(5)			
4	2	0.0005(9)	5	2	0.0005(8)			
4	3	-0.0003(11)	5	3	0.0009(9)			
4	4	0.0010(13)	5	4	0.0038(12)			
4	5	0.0110(13)	5	5	0.0290(15)			

Table 4: Overlaps of Polyakov loops in the k=2A and k=2S representations, with blocking levels b_A and b_S respectively. In SU(5) at $\beta = 80.0$ on a $24^2 32$ lattice.

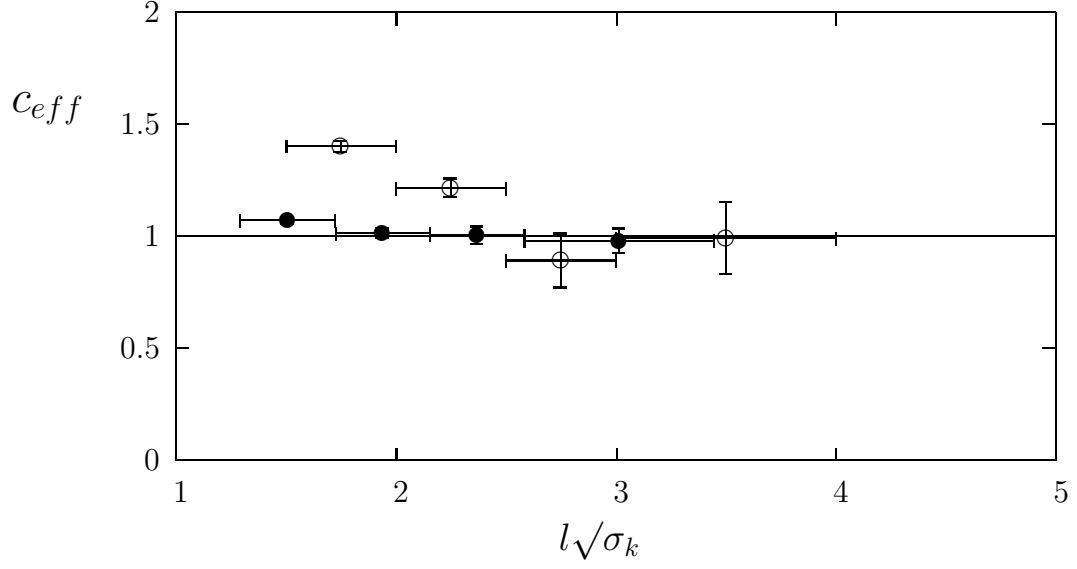


Figure 1: The effective coefficient of the $\pi/3\sigma l^2$ term in the Nambu-Goto expression for the ground state energy, eqn(6), for the range of lengths indicated. For $k = 1$, \bullet , and $k = 2$, \circ , strings in SU(4) at $\beta = 32.0$.

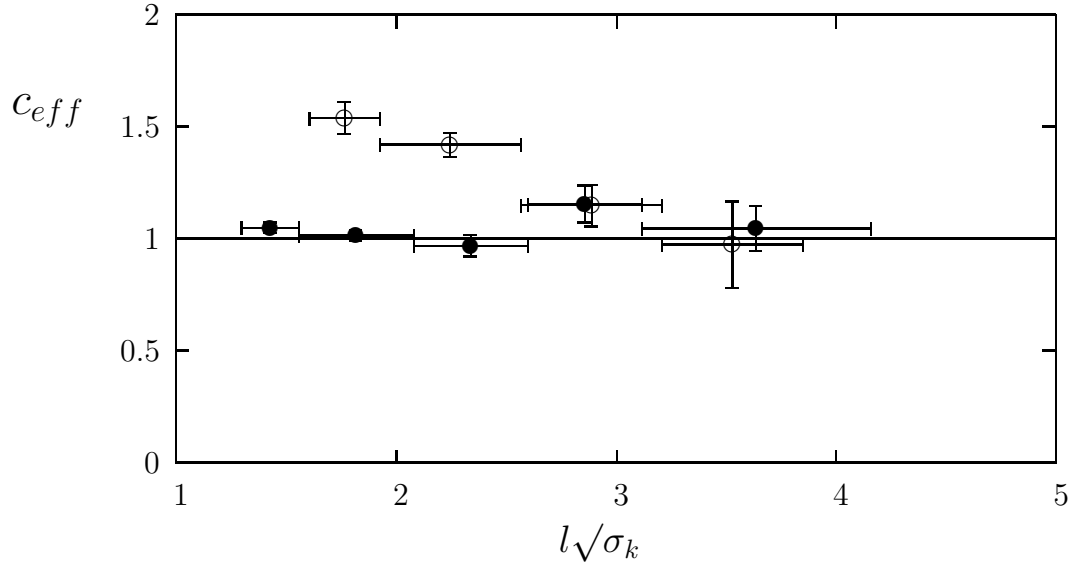


Figure 2: As in Fig.1 but for SU(5) at $\beta = 80.0$.

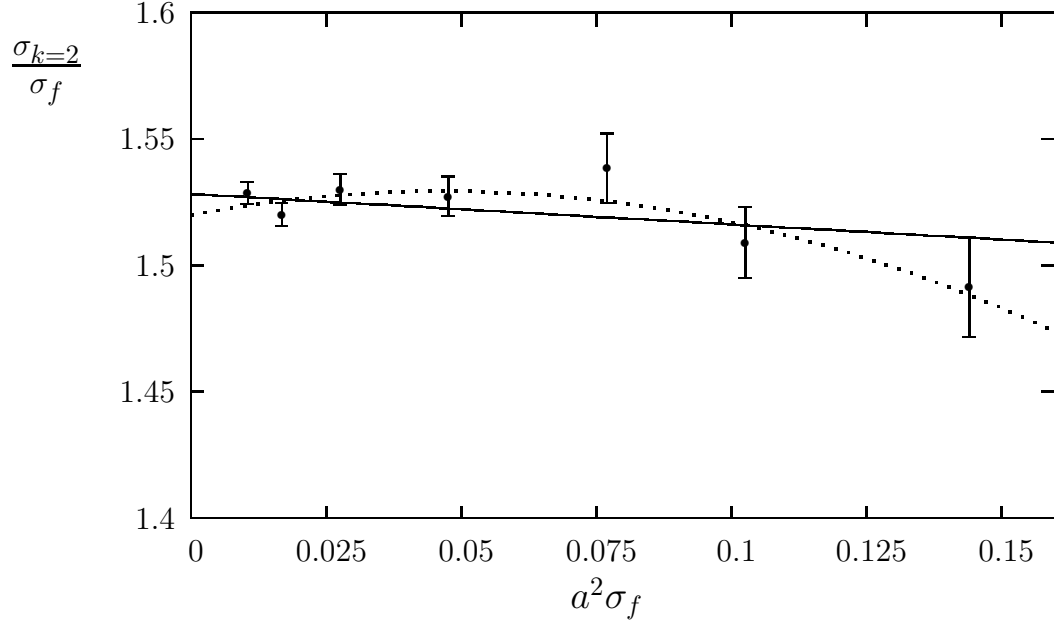


Figure 3: $k = 2$ string tension in SU(5), extracted using single exponential, S , fits, as a function of the lattice spacing. Continuum extrapolations linear and quadratic in a^2 are shown.

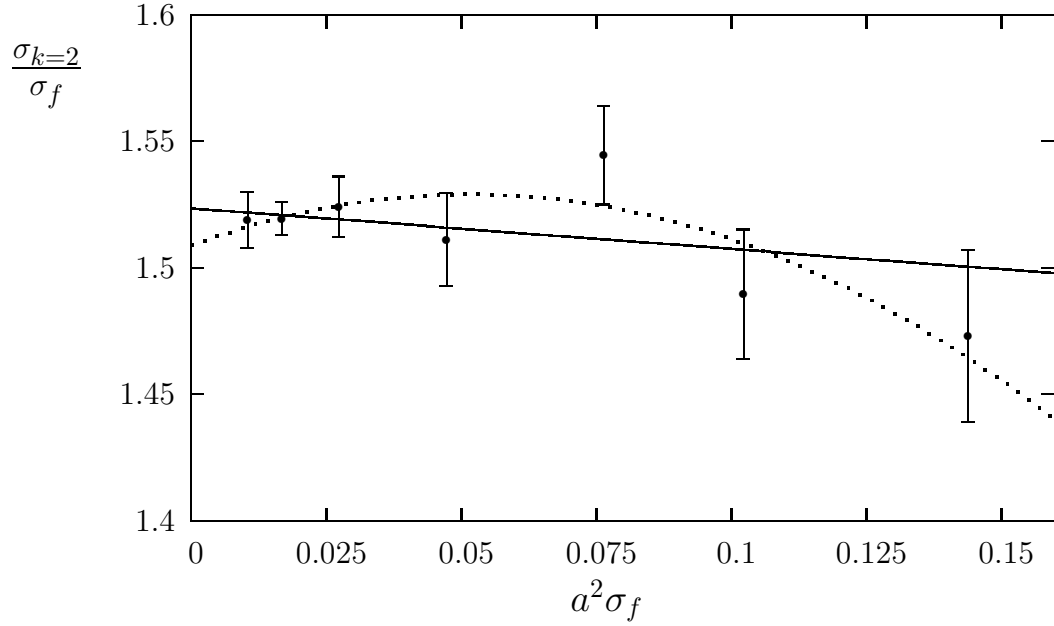


Figure 4: As in Fig.3, but using double exponential, D , fits.

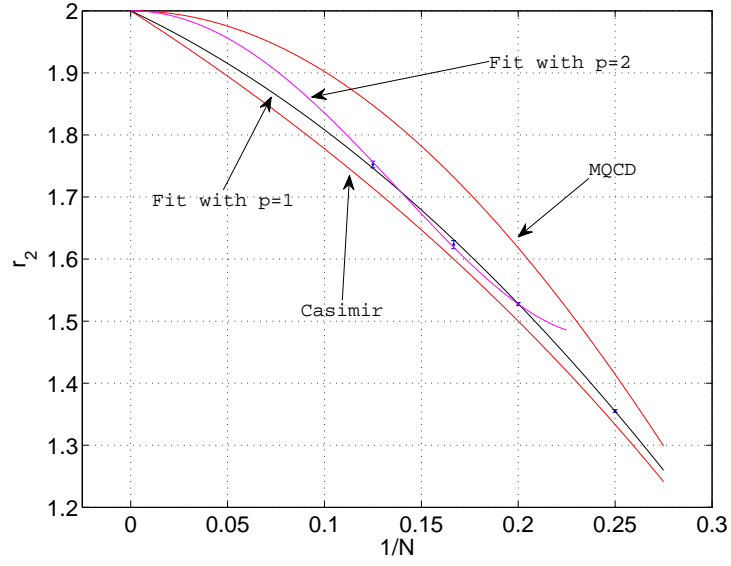


Figure 5: Calculated values of $r_2 = \sigma_{k=2}/\sigma_f$ and fits as discussed in Section 5.

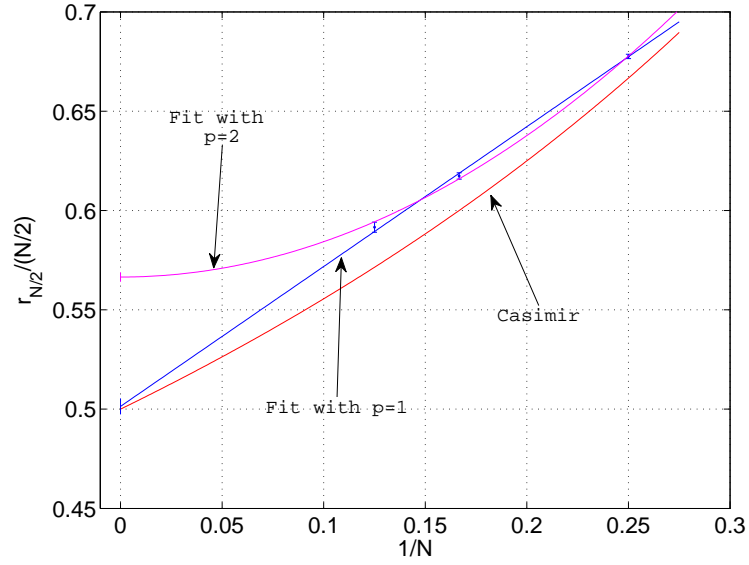


Figure 6: Calculated values of $r_{N/2} = \sigma_{k=N/2}/\sigma_f$ and fits as discussed in Section 5.

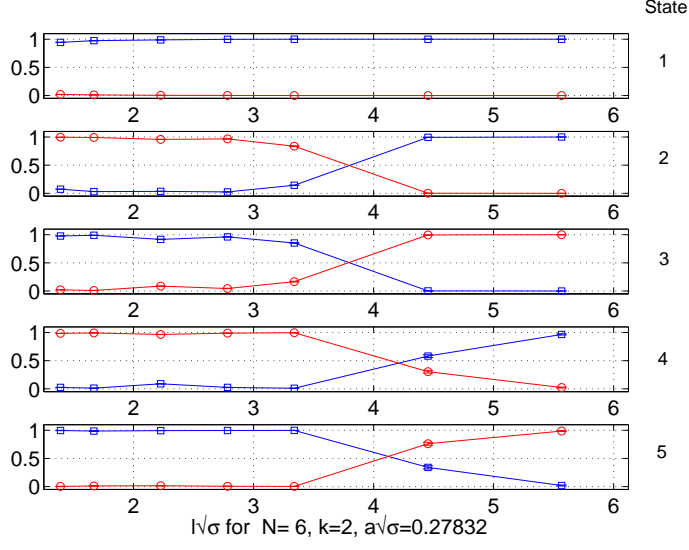


Figure 7: Overlaps of the five lowest states in the $k = 2$ spectrum of $SU(6)$ at $\beta = 59.40$ ($a \simeq 0.12$ fm) vs. the string length. In blue(red) are the overlaps onto the antisymmetric(symmetric) representation.

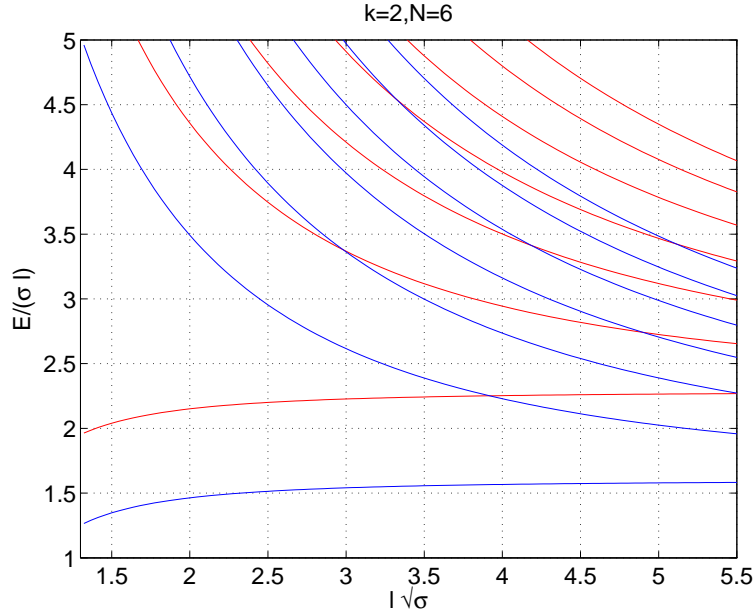


Figure 8: The spectrum of energies vs. the string length from the Nambu-Goto model for $SU(6)$ and $k = 2$. Blue(red) denotes the energies of the antisymmetric(symmetric) representation.